Chapter three
Dynamics & Deformable Bodies

Just as all of statics reduced to two equations that equaled zero, dynamics reduces to two equations. These are not equal to zero, but give the resulting acceleration. In words, again, if you apply a net force or torque to a body, it will accelerate in the direction of that force or torque. The first statement, should be familiar as Newton's law, the second equation is the angular equivalent where **α** (alpha) is the angular acceleration.

$$\sum \vec{F} = m \cdot \vec{a}$$

$$\sum \vec{\Gamma} = I \cdot \vec{\alpha}$$

Since velocity has direction as well as magnitude, *any rotating body is accelerating, even if it is running at a constant number of revolutions per time interval*. We have special names for angular velocity and acceleration; velocity is denoted by **ω** (omega) and acceleration by α (alpha). As in the linear case, the velocity is the change of position with time, and acceleration the change of velocity with time. Velocity is usually expressed in angular terms, like degrees per second or radians per second; acceleration as degrees or radians per second squared. A point on a motor shaft rotating at a constant number of RPMs has a constant acceleration, but it isn't zero! The angle is constantly changing, and with it the direction of the velocity.

The rotational equivalent of mass is **I**, the polar moment of inertia. This is a measure of the total mass in the rotating body and its distribution around the center of rotation. For example, the moment of inertia for a disk rotating about its center like a phonograph record on its turntable is $1/2mr^2$ where r is the radius. A wheel with long thin spokes and a massive rim can have a value for **I** similar to that of a solid wheel of similar diameter. **I** is typically derived for complex bodies from the combining of several moments of inertia for simpler shapes, in a manner similar to that used to determine COG above.

In general, you don't think of telescopes as being under acceleration, but there are a couple of areas where this will help us. The most important of these is vibration. To see how something can be designed not to vibrate excessively, look at something that never stops vibrating. Consider a spring horizontally mounted to a wall with a weight at its end as in figure 1.

To simplify the discussion for now, assume that there is no friction between the block mass and the horizontal surface. We'll further assume that the spring is perfectly modeled by Hooke's Law, **F** = -k**x**. We encountered this law very early in the discussion of forces and vectors; k in this equation is the spring constant, and the equation tells you how much force is required to compress or stretch a spring the distance **x**. It also tells you how hard the spring pushes back when you change its length. The constant is stated in pounds/inch or Newtons/meter.

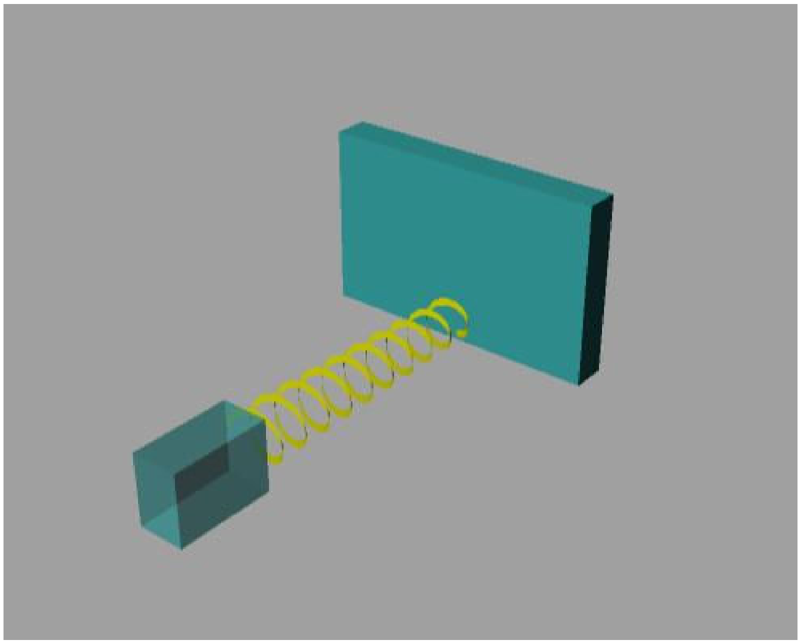

Figure 1 - A Spring/Mass System

Consider a spring anchored to a wall and a block as shown in Figure 1. It simplifies the problem if you imagine them on a frictionless surface with no side-to-side displacement as any of this happens. If we pull the block slightly to the left, the spring creates a restoring force back to the right. When you release the block it will accelerate to the right. It picks up speed until it reaches the position it was pulled from initially. At this point it has maximum speed and momentum so the block doesn't stop, but continues. This motion, however, creates a restoring force back to the left that slows the block down until it stops at a distance to the right of the starting point equal to the distance it was displaced to the left. Of course, the restoring force causes it to accelerate to the left and the spring always has a restoring force in the direction that keeps the mass moving. In fact, ignoring friction in the spring and on the surface, the system will oscillate back and forth forever.

The motion of the block is referred to as simple harmonic motion, and the equation that describes its position vs. time is a sine wave (sinusoid). The acceleration and velocity are also sinusoids, related in such a way that the velocity is at its minimum when displacement and acceleration are their largest. Mathematically, we write:

$$X = A \cdot \cos(\omega t)$$

$$v = -\omega \cdot A \cdot \sin(\omega t)$$
$$a = -\omega^2 \cdot A \cdot \cos(\omega t)$$

Where X is the position vs. time, A is the amplitude or largest distance the block is displaced, t is the time and **ω** is the angular frequency of the oscillation, 2*π*f. These equations show the velocity *v* and the acceleration *a* as a function of the angular frequency and time. Angular frequency ω is related to the spring constant k and the mass of the block:

$$\omega = \sqrt{\frac{k}{m}}$$

None of these equations will be used in later work, but are presented for their general interest.

The addition of some friction into the system changes the response like the addition of shock absorbers to a car's suspension. As you probably know, the shock absorbers damp out the spring's tendency to keep oscillating forever.

The damping characteristics can be such that the spring responds very slowly and doesn't oscillate at all. If you displace it, the block slowly returns to the starting point. If the damping is the other extreme (too little), the block oscillates for a long time. These conditions are called overdamped and underdamped, respectively. If the spring returns with just a small overshoot to its original position, it is called critically damped. It can be shown mathematically that the critically damped system will return to its final position fastest. Mathematically the damping factor usually denoted by the Greek letter zeta, ζ, falls into three ranges: in overdamped cases, $\zeta > 1$; when underdamped, $\zeta << 1$ (much less than 1), and in the critically damped case, $\zeta = 0.7071$. The mathematical forms of the equations that describe this motion are not very pretty and I won't present them here. (In fact, if you understand the equations, you don't need this chapter!)

This is illustrated in **Figure 2** which shows the settling time for an underdamped, critically damped, and overdamped system. The underdamped curve is the curve on the left; it has a larger swing in the positive and negative directions than any other. This keeps it from settling to the desired value. Conversely, the overdamped curve, the bottom most curve for the entire plot, takes the longest to make it up to the final value. The critically damped curve reaches the final value (1, in this graph) soonest.

Returning to the example system, replace the spring with a metal blade stuck into the wall. We intuitively know that if we give the end of the blade a snap it will vibrate; after all, isn't this like some sort of tuning fork? The frequency at which the blade oscillates depends on the material stiffness and the physical dimensions of the protruding part, especially the ratio of thickness to length. In particular, it is inversely proportional to length which is why a long blade oscillates at a lower frequency (I assume we have all played the childhood game of holding a ruler end down on a table, "twanging" it, then sliding the ruler further onto the table and listening to the pitch go up. If not, try it!)

All telescope mounts have natural resonant frequencies, as do all structures, so resonant frequencies are not necessarily troublesome. The problem is that the amplitude of the vibration tends to go up as the frequency goes down. If you have ever observed the strings of a guitar or piano, you will have noticed that the large bass strings vibrate more (higher amplitude, A) for a given loudness than the higher pitched strings. In general, you should try to achieve a high resonant frequency that damps out quickly. The high frequency is obtained by making the stiffness high and mass low

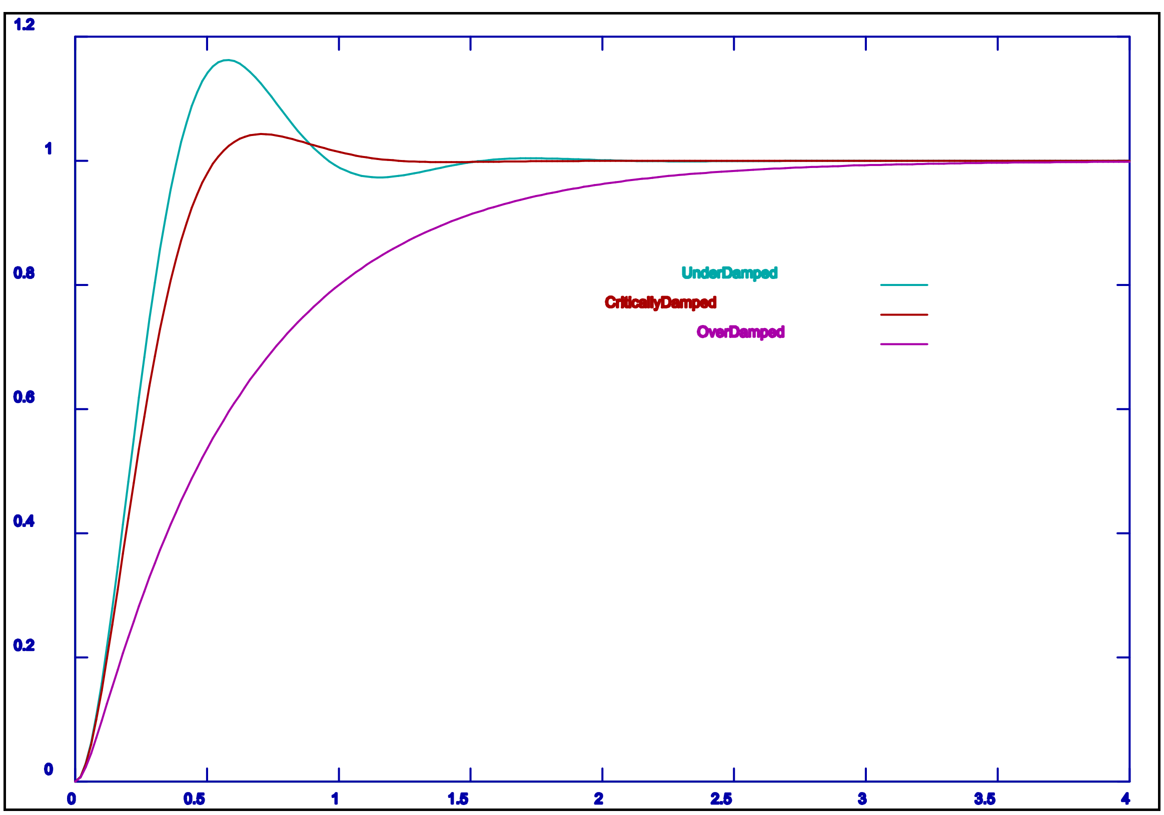

Figure 2 Response times for underdamped, overdamped and critically damped systems.

(more on this later). Damping depends more on the microstructure of the material: crystalline materials, like glass crystal or a well treated metal, will vibrate a long time; noncrystalline, random structures like plywood or Styrofoam won't. Don't forget that some woods vibrate or resonate quite well; many stringed musical instruments are made from wood.

Because of the complexity of the interrelationship of stiffness, thickness, and microstructure, I can't present exact equations to determine resonance frequency in every case. There are powerful computer models that can calculate the resonant frequencies of complex shapes and material compositions, but these are typically beyond the budget of the amateur.

We can, though, use simple equations that break down the problem by viewing the structure that's vibrating as a collection of simpler things. For example, the counterweight of a German equatorial mount is really a large mass at the end of a rod. We can model this as a point mass (i.e., one of infinitesimal size) at the end of a rod whose stiffness is determined by the material's properties and its geometry. These results are simple in that they ignore the more intricate aspects of the interactions between pieces of the mount, but they do provide a starting point. These equations will be introduced later, as we examine each geometry.

While we can't usually calculate all of the resonant frequencies of our mounts, we can measure them. This can be done by a couple of methods, but the simplest is to clamp a long, thin straight edge or ruler to the part and bang on it. A metal shop ruler 12" or longer works fine. Clamp the ruler to the mount or tube with most of it extending over the edge. Empirically, we know that the resonant frequency of a thin blade like this is:

$$f = c\frac{h}{L^2}$$

where L is the length extending beyond the edge and h is the thickness, both in inches. For a steel shop ruler c=31400; for an aluminum blade, c=31900.

The procedure is to knock on the structure with your hand hard enough to make it vibrate, but not to punch a hole in it, and note the amplitude of the vibration of the ruler. Slide the ruler further on to the part and repeat. You will find that there are lengths that give a large swing; these are resonances. Use the L value to calculate the frequency. You are aiming to find the lowest resonances present, and would like them to be as high as possible. If you find resonances evenly spaced at, say, 12, 18, and 24 Hz, suspect that there is a lower frequency oscillation (6 Hz here), and that these are harmonics of it. In general, you'd like the resonances to be as high in frequency as possible; 30 Hz or even higher.

Do you love a good stereo? You can try a modification of the method used in industry for measuring resonant frequencies. Attach a large loudspeaker with a high power amplifier to the part, and play a recording of either white noise or a frequency sweep. The ruler can be used and moved as before. Wear hearing protection. And warn the neighbors. Better yet, live somewhere with no neighbors. (The "loudspeakers" used for vibration tests in industry would amaze most stereo freaks. The "good ones" are larger than most car and small truck engines and can deliver an amazing amount of vibration).

Mechanics of Deformable Bodies

Up until this point, we have ignored any sag or length change in the things we've looked at. This can be done if the parts of the design are very strong for their load. In a lot of situations, this approximation is not valid and we need to accommodate the changes in our design.

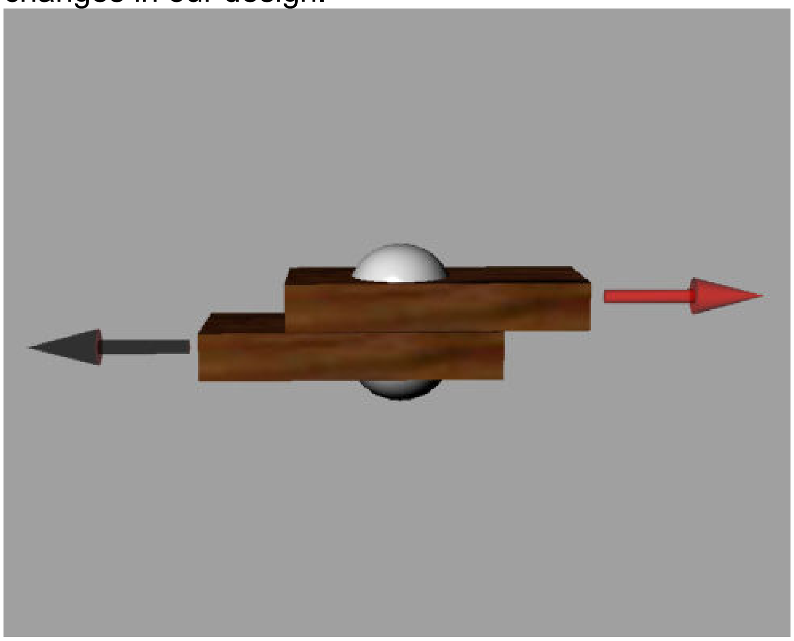

Figure 3 A Shear Plate – A bolt between two pieces of metal is under shear force from applied tensions on the two pieces.

We've already seen the concept of stress or force/unit area. This is axial or normal stress, applied along the long axis of the body such as the way you pull on a rope, or on a bolt. You can also apply stress in shear, which is perpendicular to this axis. You do this unintentionally when you fasten two objects together with bolts or rivets. If we apply force to the two plates in Figure 3, the shear stress on the bolt is:

$$\tau = \frac{F}{A}$$

where A is the cross sectional area of the bolt.

If instead of one bolt, we use two with plates on either side as in **Figure 4**, we cut the stress in half. Here,

$$\tau = \frac{F}{2A}$$

It is common practice to use these splice plates to reduce the shear on the bolts.

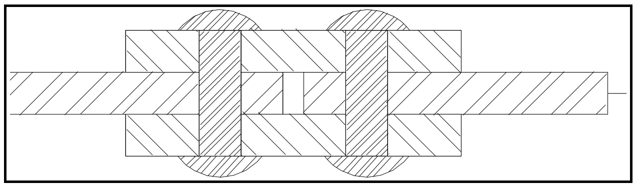

Figure 4

In reality, the distribution of forces and stresses inside an object is very complex, so we have used an average value called the bearing stress. The area used in the denominators is the product of the thickness of the plate and the diameter of the bolt. The complexity of this distribution is such that its description is mathematically quite involved, and I won't get into it here. It is interesting to note that shear cannot take place in only one plane; an equal shearing stress is exerted on another plane perpendicular to the first.

The simplest case of deformation under load is the case of the normal or axial load; you are familiar with this as an object being compressed or stretched when pushed or pulled on. In this case, the change in length, l, is:

$$l = \frac{FL}{AE}$$

where F is the force, L the original length, A the cross-sectional area and E is Young's Modulus. What if the object is made up of a series of different sections (all under the same load, along its line of action)? Then this quantity is computed for all the sections and the results added up.

For example, say we have an aluminum rod of 1.5 inch diameter that is 36 inches long. We apply a load of 100 pounds.

$$l = \frac{100\ lbs\ x\ 36\ in}{\frac{\pi}{4} x 1.5^2\ in^2 x (10 x\ 10^6 \frac{lbs}{in^2})}$$

then

$$l = 204 x\ 10^{-6}\ in$$

Because the result depends on the area of the object, a tube can be as strong as a similarly sized solid rod. This is especially true as the diameters of the tube and rod increase. If you substitute a tube of 1.5 inch outer diameter with a 1/4 inch wall thickness (i.e., it has a one inch diameter hole down its length), you will find that it lengthens 367 x 10^{-6} inch. So the solid rod lengthens 55.6% (just over half as much) of the amount the tube does.

There is a price for this; the tube contains 35.3 cubic inches of aluminum, the rod contains 63.6 cubic inches. The rod contains close to double the amount of metal and, therefore, weighs almost twice as much.

The biggest fact that you need to get out of this is that the majority of strength in a structural member is contributed by the portion farthest from the center, much like how most of your telescope's aperture is in the part of the mirror or lens farthest from the center. Again, the A in the denominator is Area, and area goes up with the square of the linear dimension. Consider a rectangular wooden beam. An undressed (full size) 4 by 4 post, for example, has an area of 16 square inches. If we used a 5 x 5 post, the area is 56.25% greater. That turns into an elongation under load of 64% of the value for the 4 x 4. To halve the elongation, increase the size of each side of the post by the square root of 2, or multiply it by roughly 1.4.

A common source of length change in structural members besides loading with weight is expansion or contraction due to temperature change. This change in length (strain) comes without the application of stress by an external agent. The length change is very easy to predict because the relationship is linear. The equation is:

$$\Delta l = \alpha(\Delta T)l$$

The change in length (delta l) is equal to a constant, denoted by the Greek letter α (alpha), times the change in temperature. Alpha is also called the Coefficient of Thermal Expansion, or CTE. Make sure whatever units you use are consistent (as always).

A classic question in freshman college or high school physics is what happens to the size of the hole in a plate as the temperature increases. The common, wrong, answer is that the hole gets smaller because the student sees the linear expansion and reasons that the distance to the nearest edge of the hole from any side of the plate gets bigger. The hole actually gets larger. It’s common in mechanics to heat a gear so that it enlarges and is easier to fit on a shaft.

The change in length (strain) can cause a stress in a member if it is constrained and can't move. This stress is:

$$\sigma = -E\alpha(\Delta T)$$

The negative sign tells us that the stress is opposite of what it could be if the part could expand or contract. If the part would ordinarily expand (positive stress due to positive delta t), the constraints must put negative stress on it to keep it from expanding. Similarly, if it were to try to contract, the supports would have to put positive stress on it to keep it from contracting. It is always opposing the strain caused by the thermal effects.

For example, assume we have our 36 inch long aluminum tube 1.5 inch diameter with 1/4 inch walls and it is mounted between two (imaginary) supports that don't change shape or move with temperature. We take it out from our 70^{o} F house on a night when the temperature is 35^{o}.

Plugging in values for E (10×10^{6}), alpha (13.1×10^{-6}) and the 35 degree change in temperature, we find the stress is:

$$\sigma = -10 * 10^{6} \frac{lbs}{in^{2}} * 13.1 * 10^{-6} \frac{1}{°F} * -35\,°F$$

$$\sigma = 4.585x\,10^{3}\ PSI$$

The force exerted on the supports is

$$F = \sigma A$$

$$F = 4585 \frac{lbs}{in^{2}} x \frac{\pi}{4} x(1.5^{2} - 1^{2})\, in^{2} = 4501 lbs$$

This 4500 pound force is due solely to the temperature change. This force is in the direction that pulls the supports together, since the tube is contracting in the cold. Note that the CTE was expressed as simply $1/^{o}$F or “per degree F”. It is often written as “inches per inch per degree F”, but the length units cancel out leaving only the temperature dependence.

Of course, any real supports would contract as well, and the resultant force would differ. This can be calculated by superposing the expansion or contraction of the various pieces. If, for example, the rod's supports contracted as well, you would calculate the amount the support changed, and then add that length change to the change in the tube.

When two different materials are attached to each other, they can exert a lot of force on each other. To see why, look at the ratio of the equations for thermal expansion:

$$\frac{\Delta L_1}{\Delta L_2} = \frac{\alpha_1 \Delta TL}{\alpha_2 \Delta TL}$$

You can see that the terms involving the initial length cancel (if they are the same length) along with the temperature change. This leaves only the material types to determine the change in length. If you fasten two pieces with different CTE's together, temperature changes can cause a lot of stress. If you've ever had hardware loosen on things exposed to temperature extremes, this is one of the reasons. It is one of the main reasons we use torque wrenches on hardware. Torquing actually stretches the bolt and thereby locks the hardware in place.

The relationships for linear strain above, and for others that will follow, assume that the force is applied uniformly on the cross section of the element we are analyzing. In compression of a rod, for example, it should be obvious that if we apply the force on a much smaller area the results can differ. Consider applying pressure to a pillow; if you put a flat piece of something over the pillow and push down in the middle of the piece with your hand, the pillow will crush evenly. Now push on the on the pillow without a plate to spread the force evenly and you'll find it indents strongly around your hand. The pillow looks very different in the two cases. Steel plates behave the same way, although the size of the deformation is drastically smaller.

Another important point is that the stress in a member is not uniform if there are holes or defects present, and the shape of the hole matters as much as its presence. A square corner is trouble and can lead to failure because stress concentrates at the corners. A round hole that is smooth after drilling has the minimum rise associated with it, and even this can double or triple the stress if it removes most of the material in the piece. In general, the larger the hole is with respect to the member it is drilled in, the more stress increases. For example, a half inch hole in a 3/4 inch wide plate doubles the stress in the material around the hole. A half inch hole in a plate 0.525 inch wide multiplies the stress by 2.5.

The same rise in stress occurs when there is an abrupt change in the area of a member as would occur when there is a small tab extending off the main body of a plate, or when a small plate is welded onto a much wider one. The relationship is more complex, but the general rule is to use a fillet to smoothly blend the two cross-sections and a more gradual blend (larger radius fillet) causes less stress concentration.

A square hole dramatically increases the stress in its vicinity; don't use a square punch. An interesting note is that in the early days of aviation, airplane makers used square cutouts for windows. It was discovered that cracks propagated away from the corners of the window and damaged the fuselage. You will notice that modern airliners all have curved windows with no sharp corners.

Bending

Pure bending in a structural member occurs when both ends of the object are subject to a force at right angles to its long axis. For example, think of pulling the string of bow and arrow set; the force on the string can be resolved into two components with one parallel to and the other perpendicular to the long axis of the bow. The same thing happens to a telescope suspended from its bearings in a mount of some sort. The force of gravity acts to curve the tube under its own weight.

Resistance to the bending force depends on the cross-sectional area in the direction of the bend. You can easily demonstrate this to yourself with a rectangular eraser like you used to have in school. The eraser bends much more easily in the

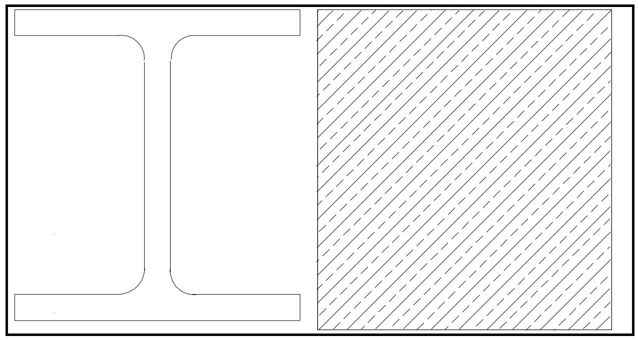

Figure 5

thin direction than in the thick direction. See **Figure 5**. This figure shows the end views of what is referred to as an “I beam”, due to its shape, and a solid rectangular beam. The I beam and the solid beam have similar resistance to bending in the direction shown here as vertical. The material in the middle of the beam is not necessary and is removed to yield the I shape. On the other hand, the I beam is much more susceptible to bending in the direction shown here as horizontal. The solid beam has more material in that direction.

If you know that the force is always going to be exerted in a given direction an I beam shape can save weight and material. When the force can be exerted from any direction, a circular cross section is best. The bottom parts of a telescope mount, where the force is always in the vertical direction, can use I beam structures. The tube, especially if it is on an equatorial mount, should use circular elements. The I beams can be wood; three 1 x 4 boards solidly joined for example, fiberglass covered foam, or any other material.

Round tubes are also best under twisting or torsion, which can occur when the tube is loaded by a weight that is off axis.

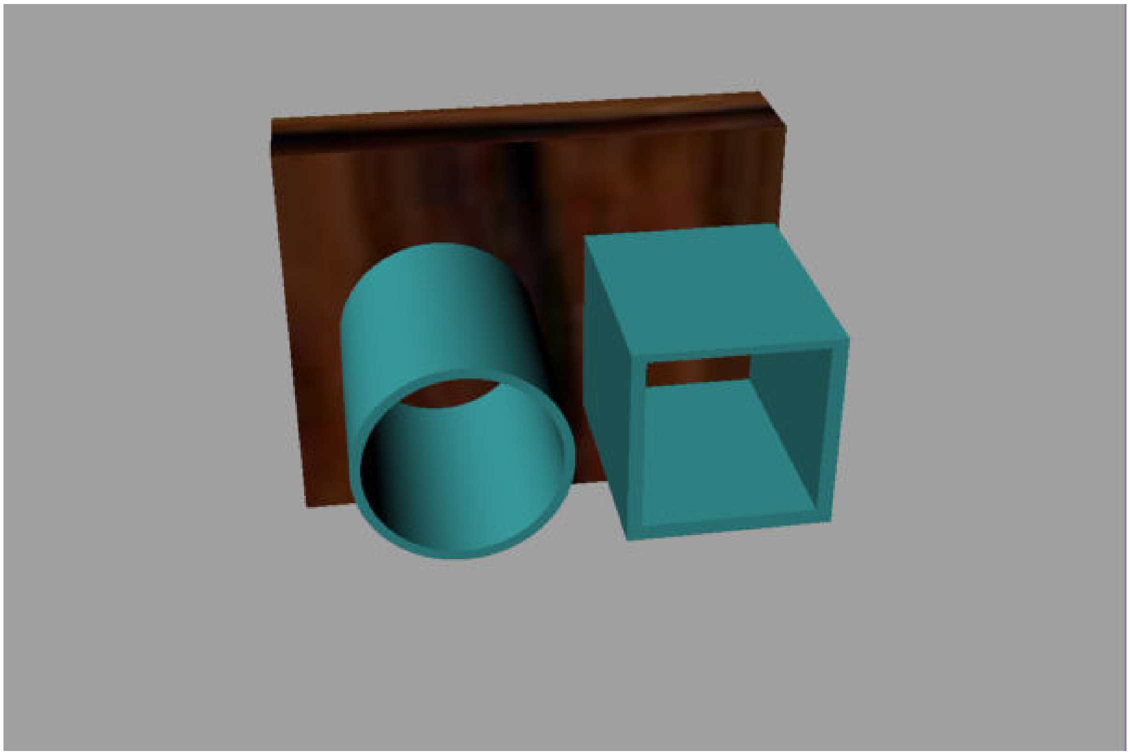

Figure 6 A circular tube can handle the same load from any direction.

Because of their usefulness, we will emphasize the properties of circular cross sections. The deflection of any member is a function of what is termed its area moment of inertia. For a circular rod:

$$I = \frac{\pi}{4} r^4 \qquad \textbf{1}$$

since r=d/2, it's often more convenient to calculate:

$$I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}\left(\frac{d^4}{16}\right) = \frac{\pi}{64} d^4 \qquad \textbf{2}$$

For a tube:

$$I = \frac{\pi}{64}\left(d_{outer}^4 - d_{inner}^4\right) \qquad \textbf{3}$$

Notice that since the diameter is raised to the fourth power, I has units of in^4 or m^4.

If you want to use a solid rectangular beam of base b and height h, the moment of inertia for bending in the height direction, i.e., about the line marked h in **Figure 7**, is:

$$I_h = \frac{b\,h^3}{12} \qquad \textbf{4}$$

about the base (the line marked b) it is:

$$I_b = \frac{b^3\,h}{12} \qquad \textbf{5}$$

In the special case of a square beam:

$$I = \frac{l^4}{12} \qquad \textbf{6}$$

where l is the length of any side.

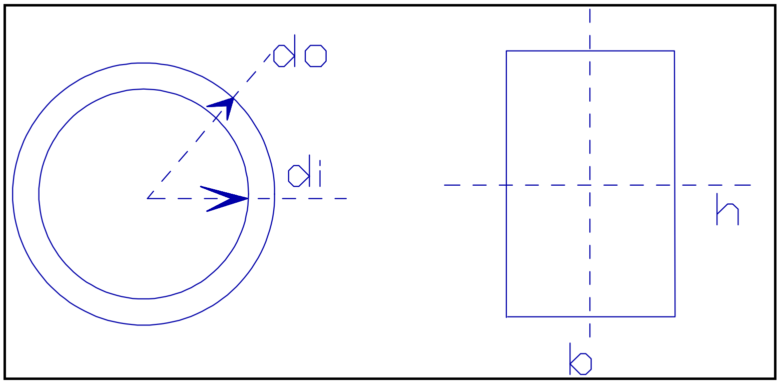

Figure 7

The sag at the end of any beam of length l subjected to load F, area moment of inertia I, elastic modulus E, and clamped at one end is:

$$S = \frac{F\,l^3}{3EI} \qquad \textbf{7}$$

The sign of this equation says that the sag is in the same direction as the load, something we should be able to conclude by ourselves! This equation is one of the most important ones we will come across. The curve that the sagging member takes is a cubic curve, so the farther we get from the cantilever (clamp) point, the more pronounced the curve is.

The centerline of the member makes an angle with its initial reference position described by:

$$\Theta = \frac{F L^2}{2EI} \qquad \textbf{8}$$

The angle theta (θ), measured in radians, can be seen to increase rapidly with increasing length of the object, since L is squared.

Consider a scope mounted in a conventional Dobsonian type mount, although the mount design is not important to the problem, as in **Figure 8**. The telescope appears to be simply supported, but the tube itself is held by a clamp arrangement (usually a box built around it), so we will treat it as being cantilevered. The load from the mirror and cell is equal to the load from the finder, eyepiece, camera and what not; it is six pounds at each end. Let's examine two cases. The first will be a 7" ID aluminum tube with a 1/8 inch wall, the second will be a square plywood tube made from 1/4" plywood. It will be 7 inches on a side, inside length.

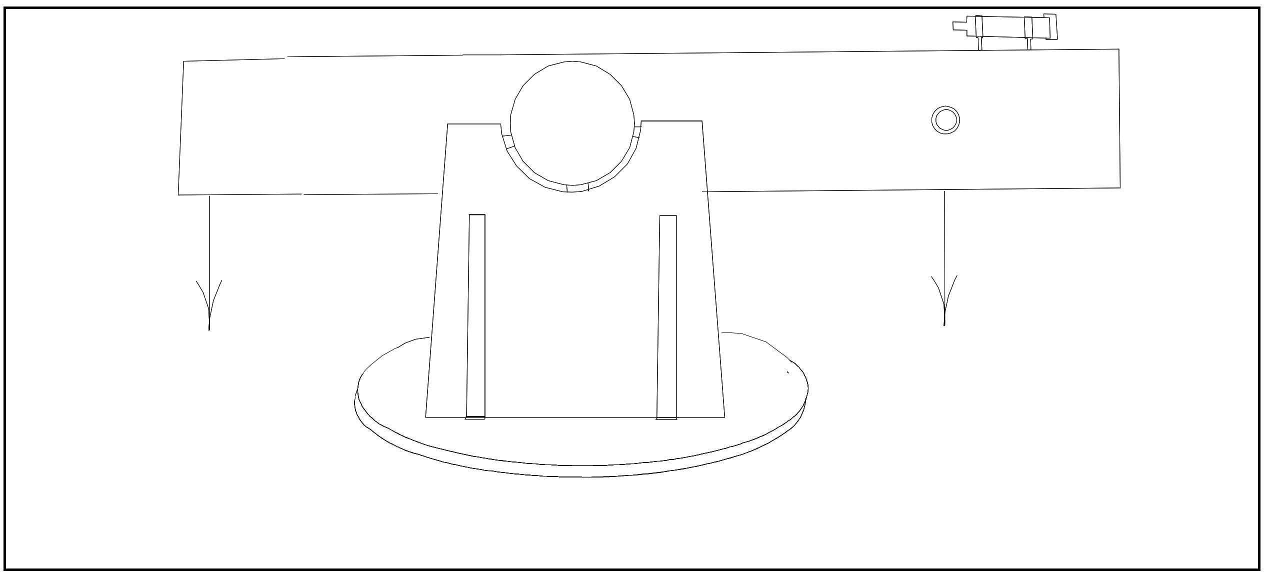

Figure 8

First we calculate I for each type tube:

$$I_{rnd} = \frac{\pi}{64}(7.25^4 - 7.00^4) = 17.76 \qquad \textbf{9}$$

$$I_{sqr} = \frac{1}{12}(7.50^4 - 7.00^4) = 63.59 \qquad \textbf{10}$$

We note that I for the square tube is much larger than for the round tube; this is because there is more material far from the center – the cut off corners in the circle.

Then:

$$S_{rnd} = \frac{6x\,24^3}{3(10x\,10^6)(17.76)} = 156x\,10^{-6}\,inch \qquad \textbf{11}$$

$$S_{sqr} = \frac{6x\,24^3}{3(1.6x\,10^6)(63.59)} = 271x\,10^{-6}\,inch \qquad \textbf{12}$$

We find that the square plywood tube sags just under twice as much as the round metal tube, despite its much higher moment of inertia. This is due to the lower modulus of elasticity for the plywood.

By substituting into equation 8, we can calculate the angular displacement of a ray in the aluminum tube deflected by this weight:

$$\Theta = \frac{6x\,24^2}{2(10x\,10^6)(17.76)} = 9.73x\,10^{-6} \qquad \textbf{13}$$

and theta (the angle) is 9.73 x 10^{-6} *radians* (in the aluminum tube). (The radian is the "natural" measure of an angle; it is unitless, and the relationship shown is valid for any units of force or length, since they cancel out.) There are 2π radians in a circle, and this sag converts to 557 x 10^{-6} degrees or, continuing the conversion:

$$557x\,10^{-6}\,degreesx60\,\frac{minutes}{degrees}\,x60\,\frac{seconds}{minute} = 2.01\,seconds \qquad \textbf{14}$$

Don't forget that this is the sag of one end of the tube and the other end sags the same amount for a total of 4 arcseconds. How much is this? The diameter of Mars as seen from earth during a good opposition is around 20 seconds.

Getting back to the discussion of pure bending, we have examined the case of bending in a symmetrical shape about an axis of symmetry. The bending takes place about what is called the neutral axis or NA of the shape. If the amount of bending is small enough that the stresses remain in the elastic range (as we want), the neutral axis passes through the centroid of the shape. The neutral axis is called this because there is no stress along it; when a bending force is applied, the material on the side of the neutral axis where the bending force is applied is under compression, the side away from the force is under tension. This is, again, the center of a symmetrical shape.

As was the case with statics, there is much here we are not covering. You can analyze the bending of rods and beams in virtually any configuration. It is frequently useful to model a uniformly distributed load. The most useful example of this is the sag of a member under its own weight, cantilevered at the end. The sag here is:

$$S = \frac{W\,L^4}{8EI} \qquad \textbf{15}$$

The angular rotation at the end of the beam deformed by the uniform load is given by:

$$\Theta = \frac{F\,L^3}{6EI} \qquad \textbf{16}$$

You can also model different supports. For instance, if you fix a beam at both ends and apply a load in the middle (the various forms of yoke mounts are an example), the maximum sag at the middle is:

$$S = \frac{F\,l^3}{48EI} \qquad \textbf{17}$$

and is (of course) in the same direction as the weight.

The matter in which the ends of the beam are held is important, too. A support that fixes the ends of the beam and prevents them from linear or rotational motion is called a cantilever (as previously mentioned). If the object is merely resting on a support that doesn't prevent it from moving, that is a simple support, and it yields equations that are different. These equations have been for cantilever ends.

Buckling

The final topic in deformable bodies that I want to examine is buckling. Perhaps you have seen something buckle; if so, you will realize that it is the sudden, seemingly unpredictable collapse of something under load. For a demonstration, roll up a piece of regular typing or notebook paper to make a tube about an inch in diameter and 11 inches long. Tape it in a couple of places just to hold it together. Now stand it on a smooth flat surface, such as a table, and start stacking cans or other (unbreakable!) weights on it. Everything goes fine up to some point and then the paper will suddenly kink and collapse. Mine took over five pounds.

The first person to study this problem and quantify it was the great mathematician Leonhard Euler (pronounced “oiler”), and he derived that the critical load that causes the member under load to buckle, P_{cr} is

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad \textbf{18}$$

Similarly, the critical stress in the member is:

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \qquad \textbf{19}$$

The quantity (L/r) is important enough to have a special name; the slenderness ratio of the element. The r used is the radius of the rod, or the smaller dimension of a rectangular or I beam, since it will bend in the smallest plane.

These equations tell us that buckling is a strong function of length; doubling the length will cut the critical axial load to 1/4 of its initial value. On the other hand, doubling the moment of inertia will double the critical axial load.

The assumptions in here are for a load placed through the center of the cross section. If you place if off center, the critical load goes down. It also assumes that both ends of the element are pinned; that is, they are free to rotate, but not to move linearly. A rod held in a cantilever at one end and pinned at the other has an effective length that is twice its actual length, and thus can support 1/4 of the load you think it can. The simple act of fastening the rod at both ends and making them completely immobile (in a cantilever), though, yields an effective length of 1/2 of the actual length. It can support 4 times what a pinned rod could support. This has the added advantage of helping to ensure that the load is centered on the rod. This is why the good designs for a truss telescope clamp the tubes at both ends. Figure 9 shows the equivalent length for three columns, all of length l.

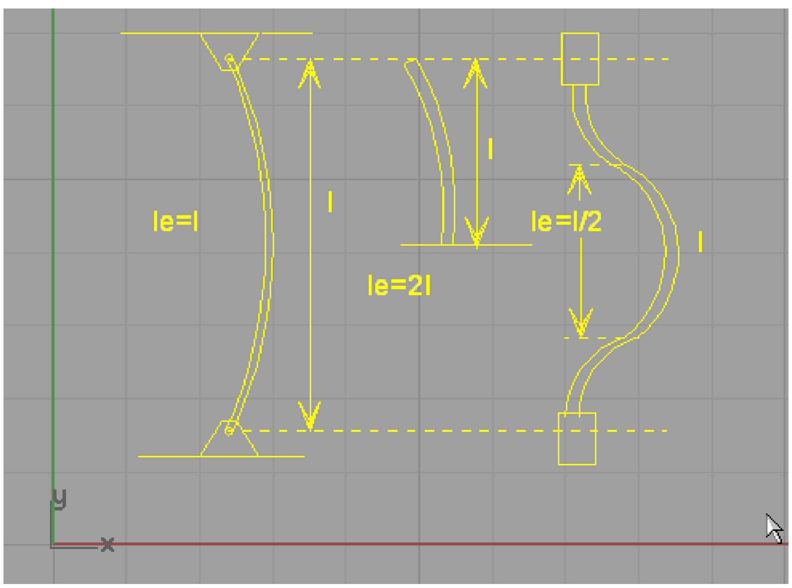

Figure 9 Critical Length of a Rod

When loading occurs off the centerline, called eccentric loading, we find that the maximum load drops dramatically. For a long, thin member, the load can be cut in half. Shorter, thicker beams have an inherently higher load capacity because of their lower slenderness ratio, and are therefore less likely to buckle.

Buckling is hard to predict and can happen catastrophically (as seen with the paper tube demo) causing equipment damage or personal injury. Furthermore, Euler's equation is an approximation and doesn't include the effects of eccentric loading. Because of this, a larger than normal factor of safety is wise here. Calculate the P_{cr} for the size beam you want to use, and opt for a larger beam if the calculated value isn't at least three times larger than the expected load. In other words, the calculated critical buckling load should be at least three times the load carried.

The factor of safety here also helps protect from failure due to a load in an unexpected direction. Do the paper tube experiment again, and use half the weight that made it collapse. Now push gently on the side of the tube with a finger. Pretty dramatic? You don't want your telescope tube to collapse if you bump into it.

Telescope tubes are loaded by the component of the weights in the tube that are along the long axis of the tube. As a rule of thumb, we want a tube with a wall thickness that is about 1 to 2% of its outer diameter, or about 1/8 to 1/4 inch for a 12 inch diameter tube. The heavier wall gives a better slenderness ratio (L/r) which reduces the tendency to buckle. In reality, we don't load tubes very heavily for their diameter so the lower figure is acceptable.

For example, say we have an aluminum tube with inner diameter 12 inches and outer diameter 12.125 inches. The moment of inertia is 43.08 in^4 and E is 10 x 10^6 PSI. Let's assume the length under load is one-half the tube length, and I'll call it 30 inches. This is the middle geometry in the figure, and the effective length is twice this, or 60 inches. The critical load before buckling is:

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (10x\,10^6)(43.08)}{60^2} = 1.18x\,10^6\,lbs \qquad \textbf{20}$$

This is the component along the tube. This is only about a million times more load than you'd normally put on a tube. It should resist buckling.

A Practical Example -- Shaft Size

Before leaving the topic of deformable bodies, let's look at how it can be used to design the shaft for a mount. The maximum stress in a shaft under bending force is:

$$\sigma = \frac{Mc}{I} \qquad \textbf{21}$$

where M is bending moment, c is the radius, and I the area moment of inertia. We frequently use solid shafts, in which case this equation can be simplified to:

$$\sigma_a = \frac{32M}{\pi d^3} \qquad \textbf{22}$$

For a tube we have to use equation 21.

How we use this in practice is to define an allowable stress in the rod and then calculate the required diameter. The allowable stress is usually a factor of two to four less than the yield stress for the material. Stated another way, we'll allow 1/2 to 1/4 of the stress that would permanently deform the shaft.

For example, let's look at the size required for a solid steel shaft to support a typical load. The yield stress given in table 2 for steel is 35 x 10^3 PSI (also referred to as 35 KSI). We'll design in a safety factor of 3.5 and allow a stress of 10 KSI. Thus:

$$d = 3\sqrt{\frac{32M}{\pi\,\sigma_a}} \qquad \textbf{23}$$

We add up the torques (moments) and find that the total M is 1200 inch-pounds. Then we solve:

$$d = 3\sqrt{\frac{32x1200}{\pi x\,10^4}} = 1.07\,inch \qquad \textbf{24}$$

This is an odd size, so we would ordinarily go to the next size larger shaft. Here, with the result so close to 1 inch and the generous safety factor, we might use the 1 inch size shaft and sacrifice a little of that factor of safety.

This leads us to another of those engineering trades we are faced with. The one inch shaft may weigh more than we want. What do we do? Although the diameter is harder to solve for when we want to use a tube, we can calculate the moment of inertia of the solid rod we just solved for, and then substitute a tube with an equivalent or larger value for I. The tube will weigh less than the solid shaft.

For example, the moment of inertia of a one inch solid rod is 0.049 in^4, and a one foot piece weighs 14.4 ounces. A 1 1/2 in. OD #18 tube has a slightly larger moment of inertia, .0589, and weighs 4.16 ounces.

How about ball bearings? Ball bearings are available from manufacturers in a variety of sizes and styles, and they provide information on how much stress they can handle without deformation. In typical use, these bearings are called on to rotate at hundreds to thousands of RPM for long periods of time; in telescope use, they may rotate at the equivalent of a few rotations per day at most. The loading we are interested in is the static loading, and the typical telescope will never come close to the static loads that they are rated for. For example, the one inch bearing called for above is typically rated at 1500 pounds of equivalent static load.

The equivalent static load is easy to calculate from an equation given in a standard mechanical engineer's handbook:

$$P = 0.6\,R + 0.5\,T$$

where R is the radial load and T the axial load on the bearings. The radial load is typically caused by torque from the rest of the mount, and the axial load is from the downhill component of the weight of the rest of the mount.

Another tradeoff in the design of shafts and bearings is that larger bearings have a smoother feel. I have read conflicting explanations of this effect and, frankly, don't know which side to believe. Suffice it to say that pretty much everyone agrees that the effect is real and a telescope mounted with a larger shaft will feel smoother than one mounted with a smaller shaft, all other factors being equal. This is not the place to skimp on your mount, even though it may well be among the most expensive places.

References and Further Reading

(In addition to the following, the interested reader can find many solved problems in the Schaum's Outline series of books from MacGraw-Hill. These are available in many public and college libraries, as well as larger bookstores and college bookstores.)

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